On the Brauer group of Enriques surfaces

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1. Introduction

Let S be a complex Enriques surface, and $\pi:\mathrm{X}\rightarrow\mathrm{S}$ its 2-to-1 cover by a K3 surface. Poincaré duality provides an isomorphism $\mathrm{H}^3(\mathrm{S},\mathbb{Z})\cong\mathrm{H}_1(\mathrm{S},\mathbb{Z})=\mathbb{Z}/2$, so that there is a unique nontrivial element b_{S} in the Brauer group $\mathrm{Br}(\mathrm{S})$. What is the pull-back of this element in $\mathrm{Br}(\mathrm{X})$? Is it nonzero?

The answer to the first question is easy in terms of the canonical isomorphism $\mathrm{Br}(\mathrm{X})\xrightarrow{\sim}\mathrm{Hom}(\mathrm{T}_{\mathrm{X}},\mathbb{Q}/\mathbb{Z})$ (see § 2): π^*b_{S} corresponds to the linear form $\tau\mapsto(\beta\cdot\pi_*\tau)$, where β is any element of $\mathrm{H}^2(\mathrm{S},\mathbb{Z}/2)$ which does not come from $\mathrm{H}^2(\mathrm{S},\mathbb{Z})$. The second question turns out to be more subtle: the answer depends on the surface. We will characterize the surfaces S for which $\pi^*b_{\mathrm{S}}=0$ (Corollary 5.7), and show that they form a countable union of hypersurfaces in the moduli space of Enriques surfaces (Corollary 6.5).

Part of our results hold over any algebraically closed field, and also in a more general set-up (see Proposition 4.1 below); for the last part, however, we need in a crucial way Horikawa's description of the moduli space by transcendental methods.

The question considered here is mentioned in [H-S], Cor. 2.8. I am indebted to J.-L. Colliot-Thélène for explaining it to me, and for very useful discussions and comments. I am grateful to J. Lannes for providing the topological proof of Lemma 5.4.

2. The Brauer group of a surface

Let S be a smooth projective variety over a field; we define the Brauer group $\mathrm{Br}(\mathrm{S})$ as the étale cohomology group $\mathrm{H}^2_{\text{ét}}(\mathrm{S},\mathbb{G}_m)$. This definition coincides with that of Grothendieck [G] by a result of Gabber, which we will not need here.

In this section we assume that S is a complex surface; we recall the description of $\mathrm{Br}(\mathrm{S})$ in that case – this is classical but not so easy to find in the literature. The Kummer exact sequence

$$0\rightarrow\mathbb{Z}/n\longrightarrow\mathbb{G}_m\longrightarrow\mathbb{G}_m\rightarrow 0$$

gives rise to an exact sequence

$$0\rightarrow\mathrm{Pic}(\mathrm{S})\otimes\mathbb{Z}/n\longrightarrow\mathrm{H}^2(\mathrm{S},\mathbb{Z}/n)\xrightarrow{p}\mathrm{Br}(\mathrm{S})[n]\rightarrow 0\qquad(2.a)$$

(we denote by $\mathrm{M}[n]$ the kernel of the multiplication by n in a $\mathbb{Z}$-module M).

On the other hand, the cohomology exact sequence associated to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ gives:

$$0 \to \mathrm{H}^2(\mathrm{S},\mathbb{Z}) \otimes \mathbb{Z}/n \longrightarrow \mathrm{H}^2(\mathrm{S},\mathbb{Z}/n) \longrightarrow \mathrm{H}^3(\mathrm{S},\mathbb{Z})[n] \to 0 \tag{2.b}$$

Comparing $(2.a)$ and $(2.b)$ we get an exact sequence

$$0 \to \mathrm{Pic}(\mathrm{S}) \otimes \mathbb{Z}/n \longrightarrow \mathrm{H}^2(\mathrm{S},\mathbb{Z}) \otimes \mathbb{Z}/n \longrightarrow \mathrm{Br}(\mathrm{S})[n] \longrightarrow \mathrm{H}^3(\mathrm{S},\mathbb{Z})[n] \to 0\ . \tag{2.c}$$

Let $\mathrm{H}^2(\mathrm{S},\mathbb{Z})_{\mathrm{tf}}$ be the quotient of $\mathrm{H}^2(\mathrm{S},\mathbb{Z})$ by its torsion subgroup; the cup-product induces a perfect pairing on $\mathrm{H}^2(\mathrm{S},\mathbb{Z})_{\mathrm{tf}}$. We denote by $\mathrm{T_S} \subset \mathrm{H}^2(\mathrm{S},\mathbb{Z})_{\mathrm{tf}}$ the *transcendental lattice*, that is, the orthogonal of the image of $\mathrm{Pic}(\mathrm{S})$. We have an exact sequence

$$\mathrm{Pic}(\mathrm{S}) \xrightarrow{c_1} \mathrm{H}^2(\mathrm{S},\mathbb{Z}) \xrightarrow{u} \mathrm{T}_\mathrm{S}^* \to 0$$

where u associates to $\alpha \in \mathrm{H}^2(\mathrm{S},\mathbb{Z})$ the cup-product with α . Taking tensor product with $\mathbb{Z}/n$ and comparing with $(2.c)$, we get an exact sequence

$$0 \to \mathrm{Hom}(\mathrm{T_S},\mathbb{Z}/n) \longrightarrow \mathrm{Br}(\mathrm{S})[n] \longrightarrow \mathrm{H}^3(\mathrm{S},\mathbb{Z})[n] \to 0\ ; \tag{2.d}$$

or, passing to the direct limit over n ,

$$0 \to \mathrm{Hom}(\mathrm{T_S},\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Br}(\mathrm{S}) \longrightarrow \mathrm{Tors}\,\mathrm{H}^3(\mathrm{S},\mathbb{Z}) \to 0\ . \tag{2.e}$$

3. Topology of Enriques surfaces

(3.1) Let S be an Enriques surface (over $\mathbb{C}$). We first recall some elementary facts on the topology of S . A general reference is [BHPV], ch. VIII.

The torsion subgroup of $\mathrm{H}^2(\mathrm{S},\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$; its nonzero element is the canonical class $\mathrm{K_S}$. Let k_S denote the image of $\mathrm{K_S}$ in $\mathrm{H}^2(\mathrm{S},\mathbb{Z}/2)$. The universal coefficient theorem together with Poincaré duality gives an exact sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{k_\mathrm{S}} \mathrm{H}^2(\mathrm{S},\mathbb{Z}/2) \xrightarrow{v_\mathrm{S}} \mathrm{Hom}(\mathrm{H}^2(\mathrm{S},\mathbb{Z}),\mathbb{Z}/2) \to 0 \tag{3.a}$$

where v_S is deduced from the cup-product.

(3.2) The linear form $\alpha \mapsto (k_\mathrm{S} \cdot \alpha)$ on $\mathrm{H}^2(\mathrm{S},\mathbb{Z}/2)$ vanishes on the image of $\mathrm{H}^2(\mathrm{S},\mathbb{Z})$, hence coincides with the map $\mathrm{H}^2(\mathrm{S},\mathbb{Z}/2) \to \mathrm{H}^3(\mathrm{S},\mathbb{Z}) = \mathbb{Z}/2$ from the exact

sequence $(2.b)$. Note that $k_{\rm S}$ is the second Stiefel-Whitney class $w_2({\rm S})$; in particular, we have $(k_{\rm S}\cdot\alpha)=\alpha^2$ for all $\alpha\in {\rm H}^2({\rm S},\mathbb{Z}/2)$ (Wu formula, see [M-S]).

(3.3) The map $c_1:{\rm Pic(S)}\rightarrow {\rm H}^2({\rm S},\mathbb{Z})$ is an isomorphism, hence $(2.e)$ provides an isomorphism ${\rm Br(S)}\xrightarrow{\sim}{\rm Tors\,H}^3({\rm S},\mathbb{Z})\cong\mathbb{Z}/2$. We will denote by $b_{\rm S}$ the nonzero element of ${\rm Br(S)}$.

Let $\pi:{\rm X}\rightarrow{\rm S}$ be the 2-to-1 cover of S by a K3 surface. The aim of this note is to study the pull-back $\pi^*b_{\rm S}$ in ${\rm Br(X)}$.

Proposition 3.4.– *The class $\pi^*b_{\rm S}$ is represented, through the isomorphism ${\rm Br(X)}\xrightarrow{\sim}{\rm Hom}({\rm T_X},\mathbb{Q}/\mathbb{Z})$, by the linear form $\tau\mapsto(\beta\cdot\pi_*\bar\tau)$, where $\bar\tau$ is the image of τ in ${\rm H}^2({\rm X},\mathbb{Z}/2)$ and β any element of ${\rm H}^2({\rm S},\mathbb{Z}/2)$ which does not come from ${\rm H}^2({\rm S},\mathbb{Z})$.*

Proof: Let β be an element of ${\rm H}^2({\rm S},\mathbb{Z}/2)$ which does not come from ${\rm H}^2({\rm S},\mathbb{Z})$, so that $p(\beta)=b_{\rm S}$ $(2.a)$. The pull-back $\pi^*b_{\rm S}\in{\rm Br(X)}$ is represented by $\pi^*\beta\in{\rm H}^2({\rm X},\mathbb{Z}/2)$ $\cong{\rm H}^2({\rm X},\mathbb{Z})\otimes\mathbb{Z}/2$; its image in ${\rm Hom}({\rm T_X},\mathbb{Z}/2)$ is the linear form $\tau\mapsto(\pi^*\beta\cdot\bar\tau)$. Since $(\pi^*\beta\cdot\bar\tau)=(\beta\cdot\pi_*\bar\tau)$, the Proposition follows. ∎

Part (i) of the following Proposition shows that the class $\pi^*\beta\in{\rm H}^2({\rm X},\mathbb{Z}/2)$ which appears above is nonzero. This does *not* say that $\pi^*b_{\rm S}$ is nonzero, as $\pi^*\beta$ could come from a class in ${\rm Pic(X)}$ – see § 6.

Proposition 3.5.– (i) *The kernel of $\pi^*:{\rm H}^2({\rm S},\mathbb{Z}/2)\rightarrow{\rm H}^2({\rm X},\mathbb{Z}/2)$ is $\{0,k_{\rm S}\}$.*

(ii) *The Gysin map $\pi_*:{\rm H}^2({\rm X},\mathbb{Z})\rightarrow{\rm H}^2({\rm S},\mathbb{Z})$ is surjective.*

Proof: To prove (i) we use the Hochschild-Serre spectral sequence :

$$E_2^{p,q}={\rm H}^p(\mathbb{Z}/2,{\rm H}^q({\rm X},\mathbb{Z}/2))\ \Rightarrow\ {\rm H}^{p+q}({\rm S},\mathbb{Z}/2)\ .$$

We have $E_2^{1,1}=0$, and $E_\infty^{2,0}=E_2^{2,0}={\rm H}^2(\mathbb{Z}/2,\mathbb{Z}/2)=\mathbb{Z}/2$. Thus the kernel of $\pi^*:$ ${\rm H}^2({\rm S},\mathbb{Z}/2)\rightarrow{\rm H}^2({\rm X},\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$. Since it contains $k_{\rm S}$, it is equal to $\{0,k_{\rm S}\}$.

Let us prove (ii). Because of the formula $\pi_*\pi^*\alpha=2\alpha$, the cokernel of $\pi_*:$ ${\rm H}^2({\rm X},\mathbb{Z})\rightarrow{\rm H}^2({\rm S},\mathbb{Z})$ is a $(\mathbb{Z}/2)$-vector space; therefore it suffices to prove that the transpose map

$${}^t\pi_*:{\rm Hom}({\rm H}^2({\rm S},\mathbb{Z}),\mathbb{Z}/2)\longrightarrow{\rm Hom}({\rm H}^2({\rm X},\mathbb{Z}),\mathbb{Z}/2)$$

is injective. This is implied by the commutative diagram

$$\begin{tikzcd}
{\rm H}^2({\rm S},\mathbb{Z}/2) \arrow[r, two heads, "v_{\rm S}"'] \arrow[d, "\pi^*"] & {\rm Hom}({\rm H}^2({\rm S},\mathbb{Z}),\mathbb{Z}/2) \arrow[d, "{}^t\pi_*"'] \\
{\rm H}^2({\rm X},\mathbb{Z}/2) \arrow[r, "\sim", "v_{\rm X}"'] & {\rm Hom}({\rm H}^2({\rm X},\mathbb{Z}),\mathbb{Z}/2)
\end{tikzcd}$$

plus the fact that $\operatorname{Ker}\pi^* = \operatorname{Ker} v_{\mathrm{S}} = \{0, k_{\mathrm{S}}\}$ (by (i) and $(3.a)$). ■

4. Brauer groups and cyclic coverings

Proposition 4.1 .– *Let $\pi : \mathrm{X} \to \mathrm{S}$ be an étale, cyclic covering of smooth projective varieties over an algebraically closed field k . Let σ be a generator of the Galois group G of π , and let $\mathrm{Nm} : \mathrm{Pic}(\mathrm{X}) \to \mathrm{Pic}(\mathrm{S})$ be the norm homomorphism. The kernel of $\pi^* : \mathrm{Br}(\mathrm{S}) \to \mathrm{Br}(\mathrm{X})$ is canonically isomorphic to $\operatorname{Ker}\mathrm{Nm}/(1-\sigma^*)(\mathrm{Pic}(\mathrm{X}))$.*

Proof : We consider the Hochschild-Serre spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{G}, \mathrm{H}^q(\mathrm{X}, \mathbb{G}_m) \ \Rightarrow \ \mathrm{H}^{p+q}(\mathrm{S}, \mathbb{G}_m) \ .$$

Since $\mathrm{E}_2^{2,0} = \mathrm{H}^2(\mathrm{G}, k^*) = 0$, the kernel of $\pi^* : \mathrm{Br}(\mathrm{S}) \to \mathrm{Br}(\mathrm{X})$ is identified with $\mathrm{E}_\infty^{1,1} = \operatorname{Ker}\big(d_2 : \mathrm{E}_2^{1,1} \to \mathrm{E}_2^{3,0}\big)$. We have $\mathrm{E}_2^{3,0} = \mathrm{H}^3(\mathrm{G}, k^*)$; by periodicity of the cohomology of G , this group is canonically isomorphic to $\mathrm{H}^1(\mathrm{G}, k^*) = \mathrm{Hom}(\mathrm{G}, k^*)$, the character group of G , which we denote by $\widehat{\mathrm{G}}$. So we view d_2 as a map from $\mathrm{H}^1(\mathrm{G}, \mathrm{Pic}(\mathrm{X}))$ to $\widehat{\mathrm{G}}$.

Let S be the endomorphism $\mathrm{L} \mapsto \bigotimes_{g\in\mathrm{G}} g^*\mathrm{L}$ of $\mathrm{Pic}(\mathrm{X})$; recall that $\mathrm{H}^1(\mathrm{G}, \mathrm{Pic}(\mathrm{X}))$ is isomorphic to $\operatorname{Ker}\mathsf{S}/\operatorname{Im}(1-\sigma^*)$. We have $\pi^*\mathrm{Nm}(\mathrm{L}) = \mathsf{S}(\mathrm{L})$ for $\mathrm{L} \in \mathrm{Pic}(\mathrm{X})$, hence Nm maps $\operatorname{Ker}\mathsf{S}$ into $\operatorname{Ker}\pi^* \subset \mathrm{Pic}(\mathrm{S})$. Now recall that $\operatorname{Ker}\pi^*$ is canonically isomorphic to $\widehat{\mathrm{G}}$: to $\chi \in \widehat{\mathrm{G}}$ corresponds the subsheaf L_χ of $\pi_*\mathcal{O}_{\mathrm{X}}$ where G acts through the character χ . Since $\mathrm{Nm} \circ (1-\sigma^*) = 0$, the norm induces a homomorphism $\mathrm{H}^1(\mathrm{G}, \mathrm{Pic}(\mathrm{X})) \to \operatorname{Ker}\pi^* \cong \widehat{\mathrm{G}}$. The Proposition will follow from:

Lemma 4.2 .– *The map $d_2 : \mathrm{H}^1(\mathrm{G}, \mathrm{Pic}(\mathrm{X})) \to \widehat{\mathrm{G}}$ coincides with the homomorphism induced by the norm.*

Proof of the lemma : We apply the formalism of [S], Proposition 1.1, where a very close situation is considered. This Proposition, together with property (1) which follows it, tells us that d_2 is given by cup-product with the extension class in $\mathrm{Ext}^2_{\mathrm{G}}(\mathrm{Pic}(\mathrm{X}), k^*)$ of the exact sequence of G-modules

$$1 \to k^* \longrightarrow \mathrm{R}_{\mathrm{X}}^* \to \mathrm{Div}(\mathrm{X}) \to \mathrm{Pic}(\mathrm{X}) \to 0 \ ,$$

where R_{X} is the field of rational functions on X . This means that d_2 is the composition

$$\mathrm{H}^1(\mathrm{G}, \mathrm{Pic}(\mathrm{X})) \xrightarrow{\partial} \mathrm{H}^2(\mathrm{G}, \mathrm{R}_{\mathrm{X}}^*/k^*) \xrightarrow{\partial'} \mathrm{H}^3(\mathrm{G}, k^*)$$

where ∂ and ∂' are the coboundary maps associated to the short exact sequences

$$0 \to \mathrm{R}_{\mathrm{X}}^*/k^* \to \mathrm{Div}(\mathrm{X}) \to \mathrm{Pic}(\mathrm{X}) \to 0$$

and

$$0 \to k^* \to \mathrm{R}_{\mathrm{X}}^* \to \mathrm{R}_{\mathrm{X}}^*/k^* \to 0 \ .$$

Let $\lambda \in \mathrm{H}^1(\mathrm{G}, \mathrm{Pic}(\mathrm{X}))$, represented by $\mathrm{L} \in \mathrm{Pic}(\mathrm{X})$ with $\bigotimes_{g\in \mathrm{G}} g^*\mathrm{L} \cong \mathcal{O}_\mathrm{X}$. Let $\mathrm{D} \in \mathrm{Div}(\mathrm{X})$ such that $\mathrm{L} = \mathcal{O}_\mathrm{X}(\mathrm{D})$. Then $\sum_g g^*\mathrm{D}$ is the divisor of a rational function $\psi \in \mathrm{R}_\mathrm{X}^*$, whose class in $\mathrm{R}_\mathrm{X}^*/k^*$ is well-defined. This class is invariant under G, and defines the element $\partial(\lambda) \in \mathrm{H}^2(\mathrm{G}, \mathrm{R}_\mathrm{X}^*/k^*)$. Since $\mathrm{div}\ \psi$ is invariant under G, there exists a character $\chi \in \widehat{\mathrm{G}}$ such that $g^*\psi = \chi(g)\psi$ for each $g \in \mathrm{G}$. Then $d_2^{1,1}(\lambda) = \chi$ viewed as an element of $\mathrm{H}^3(\mathrm{G}, k^*) = \widehat{\mathrm{G}}$.

It remains to prove that $\mathcal{O}_\mathrm{S}(\pi_*\mathrm{D}) = \mathrm{L}_\chi$. Since $\mathrm{div}\,(\psi) = \pi^*\pi_*\mathrm{D}$, multiplication by ψ induces a global isomorphism $u : \pi^*\mathcal{O}_\mathrm{S}(\pi_*\mathrm{D}) \xrightarrow{\sim} \mathcal{O}_\mathrm{X}$. Let $\varphi \in \mathrm{R}_\mathrm{X}$ be a generator of $\mathcal{O}_\mathrm{X}(\mathrm{D})$ on an open G-invariant subset U of X. Then $\mathrm{Nm}(\varphi)$ is a generator of $\mathcal{O}_\mathrm{S}(\pi_*\mathrm{D})$ on $\pi(\mathrm{U})$, and $\pi^*\,\mathrm{Nm}(\varphi)$ is a generator of $\pi^*\mathcal{O}_\mathrm{S}(\pi_*\mathrm{D})$ on U; the function $h := \psi\, \pi^*\mathrm{Nm}(\varphi)$ on U satisfies $g^*h = \chi(g)h$ for all $g \in \mathrm{G}$. This proves that the homomorphism $u^\flat : \mathcal{O}_\mathrm{S}(\pi_*\mathrm{D}) \to \pi_*\mathcal{O}_\mathrm{X}$ deduced from u maps $\mathcal{O}_\mathrm{S}(\pi_*\mathrm{D})$ onto the subsheaf L_χ of $\pi_*\mathcal{O}_\mathrm{X}$, hence our assertion. ∎

We will need a complement of the Proposition in the complex case:

Corollary 4.3.– *Assume* $k = \mathbb{C}$, *and* $\mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}) = \mathrm{H}^2(\mathrm{S}, \mathcal{O}_\mathrm{S}) = 0$. *The following conditions are equivalent:*

(*i*) *The map* $\pi^* : \mathrm{Br}(\mathrm{S}) \to \mathrm{Br}(\mathrm{X})$ *is injective;*

(*ii*) *Every class* $\lambda = c_1(\mathrm{L}) \in \mathrm{H}^2(\mathrm{X}, \mathbb{Z})$, *with* $\mathrm{L} \in \mathrm{Pic}(\mathrm{X})$ *and* $\pi_*\lambda = 0$, *belongs to* $(1 - \sigma^*)(\mathrm{H}^2(\mathrm{X}, \mathbb{Z}))$.

Observe that the hypotheses of the Corollary are satisfied when S is a complex Enriques surfaces and $\pi : \mathrm{X} \to \mathrm{S}$ its universal cover.

Proof : By Proposition 4.1 (*i*) is equivalent to $[\mathrm{L}] = 0$ in $\mathrm{H}^1(\mathrm{G}, \mathrm{Pic}(\mathrm{X}))$ for every $\mathrm{L} \in \mathrm{Pic}(\mathrm{X})$ with $\mathrm{Nm}(\mathrm{L}) = \mathcal{O}_\mathrm{S}$, while (*ii*) means $c_1(\mathrm{L}) = 0$ in $\mathrm{H}^1(\mathrm{G}, \mathrm{H}^2(\mathrm{X}, \mathbb{Z}))$ for every such L. Therefore it suffices to prove that the map

$$\mathrm{H}^1(c_1) : \mathrm{H}^1(\mathrm{G}, \mathrm{Pic}(\mathrm{X})) \to \mathrm{H}^1(\mathrm{G}, \mathrm{H}^2(\mathrm{X}, \mathbb{Z}))$$

is injective.

Since $\mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}) = 0$ we have an exact sequence

$$0 \to \mathrm{Pic}(\mathrm{X}) \xrightarrow{c_1} \mathrm{H}^2(\mathrm{X}, \mathbb{Z}) \longrightarrow \mathrm{Q} \to 0 \qquad \text{with } \mathrm{Q} \subset \mathrm{H}^2(\mathrm{X}, \mathcal{O}_\mathrm{X})\ .$$

Since $\mathrm{H}^2(\mathrm{S}, \mathcal{O}_\mathrm{S}) = 0$, there is no nonzero invariant vector in $\mathrm{H}^2(\mathrm{X}, \mathcal{O}_\mathrm{X})$, hence in Q. Then the associated long exact sequence implies that $\mathrm{H}^1(c_1)$ is injective. ∎

5. More topology

(5.1) As in § 3, we denote by S a complex Enriques surface and by $\pi : \mathrm{X} \to \mathrm{S}$ its universal cover. Thus X is a K3 surface, with a fixed-point free involution σ such

that $\pi\circ\sigma=\pi$. We will need some more precise results on the topology of the surfaces X and S. We refer again to [BHPV], ch. VIII.

Let E be the lattice $(-\mathrm{E}_8)\oplus\mathrm{H}$, where H is the rank 2 hyperbolic lattice. Let $\mathrm{H}^2(\mathrm{S},\mathbb{Z})_{\mathrm{tf}}$ be the quotient of $\mathrm{H}^2(\mathrm{S},\mathbb{Z})$ by its torsion subgroup $\{0,\mathrm{K}_\mathrm{S}\}$. We have isomorphisms

$$\mathrm{H}^2(\mathrm{S},\mathbb{Z})_{\mathrm{tf}}\cong\mathrm{E}\qquad \mathrm{H}^2(\mathrm{X},\mathbb{Z})\cong\mathrm{E}\oplus\mathrm{E}\oplus\mathrm{H}$$

such that $\pi^*:\mathrm{H}^2(\mathrm{S},\mathbb{Z})_{\mathrm{tf}}\to\mathrm{H}^2(\mathrm{X},\mathbb{Z})$ is identified with the diagonal embedding $\delta:\mathrm{E}\hookrightarrow\mathrm{E}\oplus\mathrm{E}$, and σ^* is identified with the involution

$$\rho:(\alpha,\alpha',\beta)\mapsto(\alpha',\alpha,-\beta)\quad\text{of }\ \mathrm{E}\oplus\mathrm{E}\oplus\mathrm{H}\ .$$

(5.2) We consider now the cohomology with values in $\mathbb{Z}/2$. For a lattice M, we will write $\mathrm{M}_2:=\mathrm{M}/2\mathrm{M}$. The scalar product of M induces a product $\mathrm{M}_2\otimes\mathrm{M}_2\to\mathbb{Z}/2$; if moreover M is *even*, there is a natural quadratic form $q:\mathrm{M}_2\to\mathbb{Z}/2$ associated with that product, defined by $q(m)=\frac{1}{2}\tilde{m}^2$, where $\tilde{m}\in\mathrm{M}$ is any lift of $m\in\mathrm{M}_2$. In particular, H_2 contains a unique element ε with $q(\varepsilon)=1$: it is the class of $e+f$ where (e,f) is a hyperbolic basis of H.

Using the previous isomorphism we identify $\mathrm{H}^2(\mathrm{X},\mathbb{Z}/2)$ with $\mathrm{E}_2\oplus\mathrm{E}_2\oplus\mathrm{H}_2$.

Proposition 5.3.– *The image of* $\pi^*:\mathrm{H}^2(\mathrm{S},\mathbb{Z}/2)\to\mathrm{H}^2(\mathrm{X},\mathbb{Z}/2)$ *is* $\delta(\mathrm{E}_2)\oplus(\mathbb{Z}/2)\varepsilon$.

Proof : This image is invariant under σ^*, hence is contained in $\delta(\mathrm{E}_2)\oplus\mathrm{H}_2$; by Proposition 3.6 (i) it is 11-dimensional, hence a hyperplane in $\delta(\mathrm{E}_2)\oplus\mathrm{H}_2$, containing $\delta(\mathrm{E}_2)$ (which is spanned by the classes coming from $\mathrm{H}^2(\mathrm{S},\mathbb{Z})$). So $\pi^*\mathrm{H}^2(\mathrm{S},\mathbb{Z}/2)$ is spanned by $\delta(\mathrm{E}_2)$ and a nonzero element of H_2; it suffices to prove that this element is ε. Since the elements of $\mathrm{H}^2(\mathrm{S},\mathbb{Z}/2)$ which do not come from $\mathrm{H}^2(\mathrm{S},\mathbb{Z})$ have square 1 (3.2), this is a consequence of the following lemma. ∎

Lemma 5.4.– *For every* $\alpha\in\mathrm{H}^2(\mathrm{S},\mathbb{Z}/2)$, $q(\pi^*\alpha)=\alpha^2$.

Proof : This proof has been shown to me by J. Lannes. The key ingredient is the *Pontryagin square*, a cohomological operation

$$\mathcal{P}:\mathrm{H}^{2m}(\mathrm{M},\mathbb{Z}/2)\longrightarrow\mathrm{H}^{4m}(\mathrm{M},\mathbb{Z}/4)$$

defined for any reasonable topological space M and satisfying a number of interesting properties (see [M-T], ch. 2, exerc. 1). We will state only those we need in the case of interest for us, namely $m=2$ and M is a compact oriented 4-manifold. We identify $\mathrm{H}^4(\mathrm{M},\mathbb{Z}/4)$ with $\mathbb{Z}/4$; then $\mathcal{P}:\mathrm{H}^2(\mathrm{M},\mathbb{Z})\to\mathbb{Z}/4$ satisfies:

a) For $\alpha\in\mathrm{H}^2(\mathrm{M},\mathbb{Z}/2)$, the class of $\mathcal{P}(\alpha)$ in $\mathbb{Z}/2$ is α^2;

b) If $\alpha \in \mathrm{H}^2(\mathrm{M}, \mathbb{Z}/2)$ comes from $\tilde{\alpha} \in \mathrm{H}^2(\mathrm{M}, \mathbb{Z})$, then $\mathcal{P}(\alpha) = \tilde{\alpha}^2 \pmod{4}$. In particular, if M is a K3 surface, we have $\mathcal{P}(\alpha) = 2q(\alpha)$ in $\mathbb{Z}/4$.

Coming back to our situation, let $\alpha \in \mathrm{H}^2(\mathrm{S}, \mathbb{Z}/2)$. We have in $\mathbb{Z}/4$:

$$\begin{aligned} \mathcal{P}(\pi^*\alpha) &= 2\,\mathcal{P}(\alpha) && \text{by functoriality} \\ &= 2\,\alpha^2 && \text{by a), and} \\ \mathcal{P}(\pi^*\alpha) &= 2\,q(\pi^*\alpha) && \text{by b).} \end{aligned}$$

Comparing the two last lines gives the lemma. ∎

Corollary 5.5 .– *The kernel of* $\pi_* : \mathrm{H}_2 \to \{0, k_\mathrm{S}\}$ *is* $\{0, \varepsilon\}$ *.*

Proof : By Proposition 5.3 ε belongs to $\operatorname{Im}\pi^*$, hence $\pi_*\varepsilon = 0$. It remains to check that π_* is nonzero on $\mathrm{H}^1(\mathbb{Z}/2, \mathrm{H}^2(\mathrm{X}, \mathbb{Z})) \cong \mathrm{H}_2$. We know that there is an element $\alpha \in \mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ with $\pi_*\alpha = \mathrm{K}_\mathrm{S}$ (Prop. 3.6 (*ii*)); it belongs to $\operatorname{Ker}(1 + \sigma^*)$, hence defines an element $\bar{\alpha}$ of $\mathrm{H}^1(\mathbb{Z}/2, \mathrm{H}^2(\mathrm{X}, \mathbb{Z}))$ with $\pi_*\bar{\alpha} \neq 0$. ∎

Corollary 5.6 .– *Let* $\lambda \in \mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ *. The following conditions are equivalent:*

(*i*) $\pi_*\lambda = 0$ *and* $\lambda \notin (1 - \sigma^*)(\mathrm{H}^2(\mathrm{X}, \mathbb{Z}))$;

(*ii*) $\sigma^*\lambda = -\lambda$ *and* $\lambda^2 \equiv 2 \pmod{4}$.

Proof : Write $\lambda = (\alpha, \alpha', \beta) \in \mathrm{E} \oplus \mathrm{E} \oplus \mathrm{H}$; let $\bar{\beta}$ be the class of β in H_2. Both conditions imply $\sigma^*\lambda = -\lambda$, hence $\alpha' = -\alpha$. Since $(\alpha, -\alpha) = (1 - \sigma^*)(\alpha, 0)$ and $2\beta = (1 - \sigma^*)(\beta)$, the conditions of (*i*) are equivalent to $\pi_*\bar{\beta} = 0$ and $\bar{\beta} \neq 0$, that is, $\bar{\beta} = \varepsilon$ (Corollary 5.5). On the other hand we have $\lambda^2 = 2\alpha^2 + \beta^2 \equiv 2q(\bar{\beta}) \pmod{4}$, hence (*ii*) is also equivalent to $\bar{\beta} = \varepsilon$. ∎

We can thus rephrase Corollary 4.3 in our situation:

Corollary 5.7 .– *We have* $\pi^* b_\mathrm{S} = 0$ *if and only if there exists a line bundle* L *on* X *with* $\sigma^*\mathrm{L} = \mathrm{L}^{-1}$ *and* $c_1(\mathrm{L})^2 \equiv 2 \pmod{4}$. ∎

Remark 5.8 .– My original proof of (5.3-5) was less direct and less general, but still perhaps of some interest. The key point is to show that on H_2 q takes the value 1 exactly on the nonzero element of $\operatorname{Ker}\pi_*$, or equivalently that an element $\alpha \in \mathrm{H}_2$ with $\pi_*\alpha = k_\mathrm{S}$ satisfies $q(\alpha) = 0$. Using deformation theory (see (6.1) below), one can assume that α comes from a class in $\operatorname{Pic}(\mathrm{X})$. To conclude I applied the following lemma:

Lemma 5.9 .– *Let* L *be a line bundle on* X *with* $\operatorname{Nm}(\mathrm{L}) = \mathrm{K}_\mathrm{S}$ *. Then* $c_1(\mathrm{L})^2$ *is divisible by* 4 *.*

Proof : Consider the rank 2 vector bundle $\mathrm{E} = \pi_*(\mathrm{L})$. The norm induces a non-degenerate quadratic form $\mathrm{N} : \operatorname{Sym}^2\mathrm{E} \to \mathrm{K}_\mathrm{S}$ ([EGA2], 6.5.5). In particular, N induces an isomorphism $\mathrm{E} \xrightarrow{\sim} \mathrm{E}^* \otimes \mathrm{K}_\mathrm{S}$, and defines a pairing

$$\mathrm{H}^1(\mathrm{S}, \mathrm{E}) \otimes \mathrm{H}^1(\mathrm{S}, \mathrm{E}) \to \mathrm{H}^2(\mathrm{S}, \mathrm{K}_\mathrm{S}) \cong \mathbb{C}$$

which is alternating and non-degenerate. Thus $h^1(\mathrm{E})$ is even; since $h^0(\mathrm{E})=h^2(\mathrm{E})$ by Serre duality, $\chi(\mathrm{E})$ is even, and so is $\chi(\mathrm{L})=\chi(\mathrm{E})$. By Riemann-Roch this implies that $\frac{1}{2}c_1(\mathrm{L})^2$ is even. ∎

6. The vanishing of π^*b_{S} on the moduli space

(6.1) We briefly recall the theory of the period map for Enriques surfaces, due to Horikawa (see [BHPV], ch. VIII, or [N]). We keep the notations of (5.1). We denote by L the lattice $\mathrm{E}\oplus\mathrm{E}\oplus\mathrm{H}$, and by L^- the (-1)-eigenspace of the involution $\rho:(\alpha,\alpha',\beta)\mapsto(\alpha',\alpha,-\beta)$, that is, the submodule of elements $(\alpha,-\alpha,\beta)$.

A *marking* of the Enriques surface S is an isometry $\varphi:\mathrm{H}^2(\mathrm{X},\mathbb{Z})\to\mathrm{L}$ which conjugates σ^* to ρ. The line $\mathrm{H}^{2,0}\subset\mathrm{H}^2(\mathrm{X},\mathbb{C})$ is anti-invariant under σ^*, so its image by $\varphi_{\mathbb{C}}:\mathrm{H}^2(\mathrm{X},\mathbb{C})\to\mathrm{L}_{\mathbb{C}}$ lies in $\mathrm{L}^-_{\mathbb{C}}$. The corresponding point $[\omega]$ of $\mathbb{P}(\mathrm{L}^-_{\mathbb{C}})$ is the *period* $\wp(\mathrm{S},\varphi)$. It belongs to the domain $\Omega\subset\mathbb{P}(\mathrm{L}^-_{\mathbb{C}})$ defined by the equations

$$(\omega\cdot\omega)=0\qquad(\omega\cdot\bar{\omega})>0\qquad(\omega\cdot\lambda)\neq 0\quad\text{for all }\lambda\in\mathrm{L}^-\text{ with }\lambda^2=-2\ .$$

This is an analytic manifold, which is the moduli space for marked Enriques surfaces. To each class $\lambda\in\mathrm{L}^-$ we associate the hypersurface H_λ of Ω defined by $(\lambda\cdot\omega)=0$.

Proposition 6.2 .– *We have* $\pi^*b_{\mathrm{S}}=0$ *if and only if* $\wp(\mathrm{S},\varphi)$ *belongs to one of the hypersurfaces* H_λ *for some vector* $\lambda\in\mathrm{L}^-$ *with* $\lambda^2\equiv 2\ (\mathrm{mod.}\ 4)$.

Proof: The period point $\wp(\mathrm{S},\varphi)$ belongs to H_λ if and only if λ belongs to $c_1(\mathrm{Pic}(\mathrm{X}))$; by Corollary 5.7, this is equivalent to $\pi^*b_{\mathrm{S}}=0$. ∎

To get a complete picture we want to know which of the H_λ are really needed:

Lemma 6.3 .– *Let* λ *be a primitive element of* L^-.

(*i*) *The hypersurface* H_λ *is non-empty if and only if* $\lambda^2<-2$.

(*ii*) *If* μ *is another primitive element of* L^- *with* $\mathrm{H}_\mu=\mathrm{H}_\lambda\neq\varnothing$, *then* $\mu=\pm\lambda$.

Proof : Let W be the subset of $\mathrm{L}^-_{\mathbb{C}}$ defined by the conditions $\omega^2=0$, $\omega\cdot\bar{\omega}>0$. If we write $\omega=\alpha+i\beta$ with $\alpha,\beta\in\mathrm{L}^-_{\mathbb{R}}$, these conditions translate as $\alpha^2=\beta^2>0$, $\alpha\cdot\beta=0$. Thus $\mathrm{W}\cap\lambda^\perp\neq\varnothing$ is equivalent to the existence of a positive 2-plane in $\mathrm{L}^-_{\mathbb{R}}$ orthogonal to λ. Since L^- has signature $(2,10)$, this is also equivalent to $\lambda^2<0$.

If $\mathrm{W}\cap\lambda^\perp$ is non-empty, $\lambda^\perp$ is the only hyperplane containing it, and $\mathbb{C}\,\lambda$ is the orthogonal of $\lambda^\perp$ in L^-. Then λ and $-\lambda$ are the only primitive vectors of L^- contained in $\mathbb{C}\,\lambda$. In particular λ is determined up to sign by H_λ, which proves (*ii*).

Let us prove (*i*). We have seen that H_λ is empty for $\lambda^2\geq 0$, and also for $\lambda^2=-2$ by definition of Ω. Assume $\lambda^2<-2$ and $\mathrm{H}_\lambda=\varnothing$; then H_λ must be contained in one of the hyperplanes H_μ with $\mu^2=-2$; by (*ii*) this implies $\lambda=\pm\mu$, a contradiction. ∎

(6.4) Let Γ be the group of isometries of L^-. The group Γ acts properly discontinuously on Ω, and the quotient $\mathcal{M} = \Omega/\Gamma$ is a quasi-projective variety. The image in $\mathcal{M}$ of the period $\wp(S, \varphi)$ does not depend on the choice of φ; let us denote it by $\wp(S)$. The map $S \mapsto \wp(S)$ induces a bijection between isomorphism classes of Enriques surfaces and $\mathcal{M}$; the variety $\mathcal{M}$ is a (coarse) moduli space for Enriques surfaces.

Corollary 6.5 .– *The surfaces* S *for which* $\pi^* b_S = 0$ *form an infinite, countable union of (non-empty) hypersurfaces in the moduli space* $\mathcal{M}$ *.*

Proof : Let Λ be the set of primitive elements λ in L^- with $\lambda^2 < -2$ and $\lambda^2 \equiv 2 \pmod{4}$. For $\lambda \in \Lambda$, let $\mathcal{H}_\lambda$ be the image of H_λ in $\mathcal{M}$; the argument of [BHPV], ch. VIII, Cor. 20.7 shows that $\mathcal{H}_\lambda$ is an algebraic hypersurface in $\mathcal{M}$. By Proposition 6.2 and Lemma 6.3 the surfaces S with $\pi^*(b_S) = 0$ form the subset $\bigcup_{\lambda \in \Lambda} \mathcal{H}_\lambda$. By Lemma 6.3 (ii) we have $\mathcal{H}_\lambda = \mathcal{H}_\mu$ if and only if $\mu = \pm g\lambda$ for some element g of Γ. This implies $\lambda^2 = \mu^2$; but λ^2 can be any number of the form $-2k$ with k odd > 1 (take for instance $\lambda = e - kf$, where (e, f) is a hyperbolic basis of H), so there are infinitely many distinct hypersurfaces among the $\mathcal{H}_\lambda$. ■

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